2) Give virtual displacement by lifting either the end of the beam or internal hinge in the beam such that the beam rotates by $\delta\theta$. Sketch the deflected position.

3) The work done by force would be equal to the $F \times (r\ \delta\theta)$ where r is the location of force from the center of rotation. The work done by couple would be $M \times \delta\theta$. Use proper sign for the work done which is

i) If the direction of force and direction of displacement are same, work is positive, otherwise negative.

ii) If the direction of couple and the direction of rotation are same, work done is positive, otherwise negative.

Write virtual work equation $\sum \delta U = 0$

Cancel out $\delta\theta$ from the equation to get the unknown reaction.

Ex. 8.1 Using principle of virtual work find reactions at supports. C is an internal hinge. Refer figure (a).

Solution:

Step 1: There are two beam lengths forming a system of connected bodies, viz. Portion AC and portion CB. The internal supports of the system are a fixed support at A and a roller support at B. Portion AC and portion CB are connected by internal hinge at C. Draw the F.B.D. Refer figure (b).

Step 2: Lifting end B, such that portion CB rotates by $\delta\theta$ about C. Refer figure (c).

Using $\quad \Sigma\, \delta U = 0$

$$R_B \times (3.5\,\delta\theta) - 10 \times (1.5\,\delta\theta) = 0$$

Canceling throughout $\delta\theta$, we have

$$R_B = 4.286 \text{ kN} \uparrow$$

.... **Ans.**

Step 3: Lifting internal hinge C, such that portion AC rotates by $\delta\theta_1$ about A and portion CB rotates by $\delta\theta_2$ about B. Refer figure (d).

Here $\quad 6.5\,\delta\theta_1 = 3.5\,\delta\theta_2$

$\therefore \quad \delta\theta_2 = 1.857\,\delta\theta_1$

Note that forces V_A and R_B don't work since they remain stationary. Also force H_A does not do work since it acts $\perp$ to the displacement of point A.

Using $\quad \Sigma\, \delta U = 0$

$$M_A \times \delta\theta_1 - 8 \times (2\,\delta\theta_1) - 5 \times (5\,\delta\theta_1) - 12 \times \delta\theta_1 - 10 \times (2\,\delta\theta_2) = 0$$

$$M_A \times \delta\theta_1 - 16\,\delta\theta_1 - 25\,\delta\theta_1 - 12 \times \delta\theta_1 - 20\,(1.857\,\delta\theta_1) = 0$$

Canceling throughout $\delta\theta_1$, we have

$\therefore \quad M_A = 90.14$ kNm

$M_A = 90.14$ kNm ↺ .... **Ans.**

Note that M_A does positive work because its direction and the direction of rotation are same, while 12 kNm couple acts in clockwise direction and the rotation of the portion AC is anti-clockwise, so it does negative work.

Step 4: Lifting end A, such that portion AC rotates by $\delta\theta$ about C. Refer figure (e). Note that force H_A being $\perp$ to the displacement does not do work. Couple M_A acts in anti-clockwise direction while portion AC rotates clockwise, so M_A does negative work. Direction of 12 kNm couple and rotation of AC are both clockwise therefore 12 kNm couple does positive work.

Using $\quad \Sigma\, \delta U = 0$

$$V_A \times (6.5\,\delta\theta) - 90.14 \times \delta\theta - 8 \times (4.5\,\delta\theta) - 5 \times (1.5\,\delta\theta) + 12 \times \delta\theta = 0$$

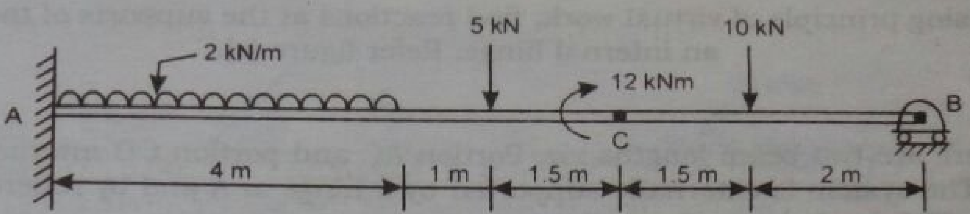
Canceling throughout $\delta\theta$, we have

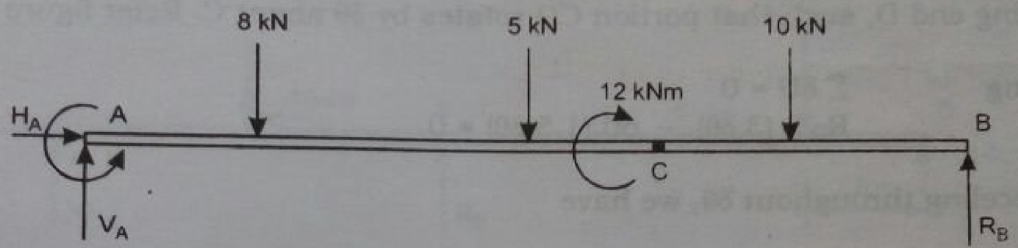
$$V_A = 18.71 \text{ kN} \uparrow$$

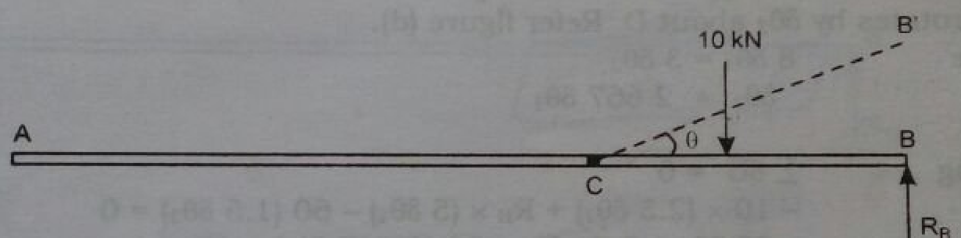
.... **Ans.**

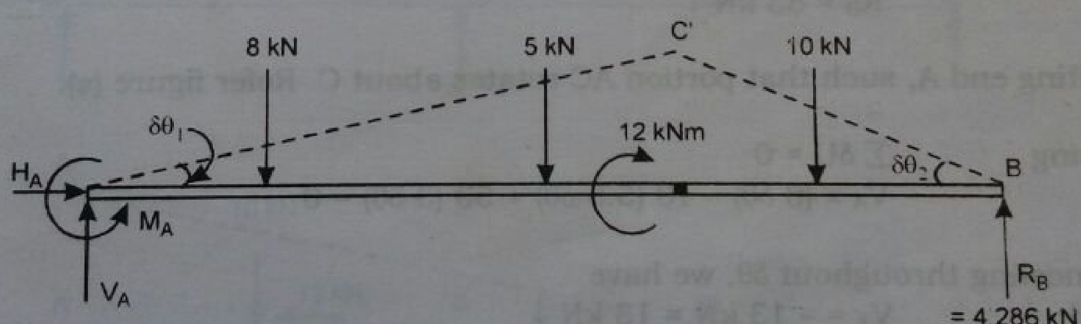
ep 5: Since there is no horizontal force or horizontal component of any force, the reaction $H_A = 0$

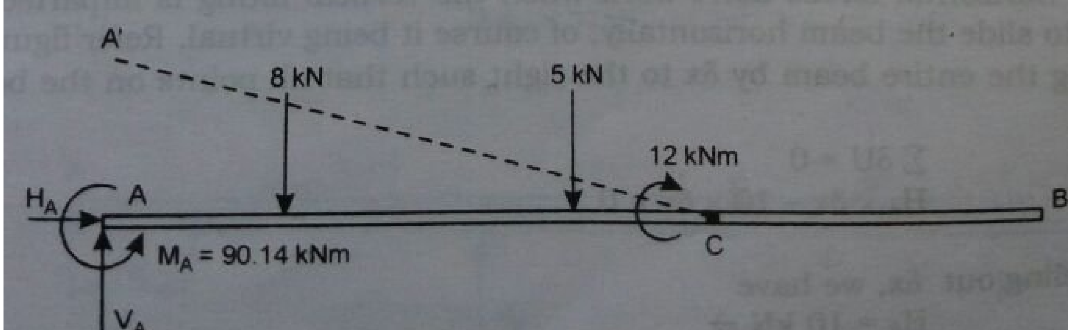
.... **Ans.**


5 kN
10 kN
2 kN/m
12 kNm
A
B
C
4 m
1 m
1.5 m
1.5 m
2 m


8 kN
5 kN
10 kN
12 kNm
H_A
A
B
C
V_A
R_B


B'
10 kN
A
θ
B
C
R_B


C'
8 kN
5 kN
10 kN
$\delta\theta_1$
12 kNm
$\delta\theta_2$
H_A
B
M_A
V_A
R_B
= 4.286 kN


A'
8 kN
5 kN
12 kNm
H_A
A
B
C
M_A = 90.14 kNm
V_A

Ex. 8.2 Using principle of virtual work, find reactions at the supports of the beam. C is an internal hinge. Refer figure (a).

Solution:

Step 1: There are two beam lengths viz. Portion AC and portion CD internally hinged at C. The system is externally supported by a hinge at A and by rollers at B and D. The FBD is shown in figure (b).

Step 2: Lifting end D, such that portion CD rotates by $\delta\theta$ about C. Refer figure (c).

Using $\Sigma\,\delta U = 0$

$R_D \times (3\,\delta\theta) - 60\,(1.5\,\delta\theta) = 0$

Canceling throughout $\delta\theta$, we have

$R_D = 30$ kN ↑ .... **Ans.**

Step 3: Lifting internal hinge C, such that portion AC rotates by $\delta\theta_1$ about A and portion CD rotates by $\delta\theta_2$ about D. Refer figure (d).

Here $8\,\delta\theta_1 = 3\,\delta\theta_2$

∴ $\delta\theta_2 = 2.667\,\delta\theta_1$

Using $\Sigma\,\delta U = 0$

$-10 \times (2.5\,\delta\theta_1) + R_B \times (5\,\delta\theta_1) - 60\,(1.5\,\delta\theta_2) = 0$

$-25\,\delta\theta_1 + 5\,R_B\,\delta\theta_1 - 90\,(2.667\,\delta\theta_1) = 0$

canceling throughout $\delta\theta_1$, we have

$R_B = 53$ kN ↑ .... **Ans.**

Step 4: Lifting end A, such that portion AC rotates about C. Refer figure (e).

Using $\Sigma\,\delta U = 0$

$V_A \times (8\,\delta\theta) - 10\,(5.5\,\delta\theta) + 53\,(3\,\delta\theta) = 0$

Canceling throughout $\delta\theta$, we have

$V_A = -13$ kN $= 13$ kN ↓ .... **Ans.**

Step 5: Since horizontal forces don't work when the vertical lifting is imparted, we will have to slide the beam horizontally, of course it being virtual. Refer figure (f). Sliding the entire beam by δx to the right such that all points on the beam shift by δx

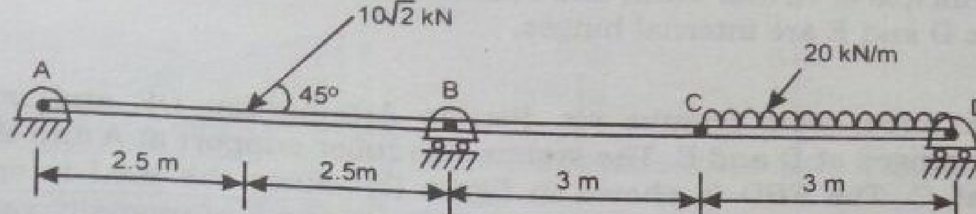
Using $\Sigma\,\delta U = 0$

$H_A \times \delta x - 10 \times \delta x = 0$

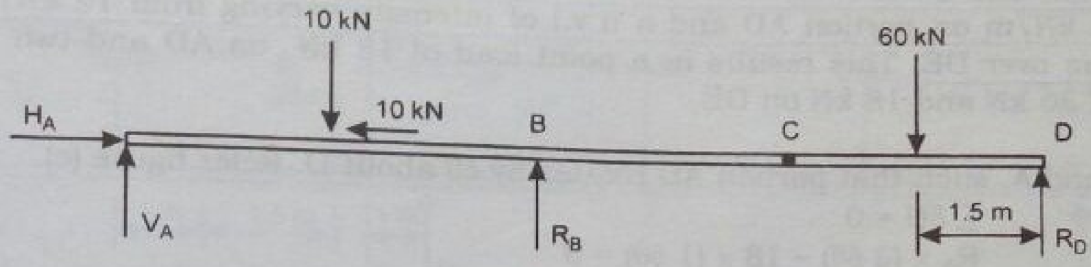
Canceling out δx, we have

$H_A = 10$ kN → .... **Ans.**

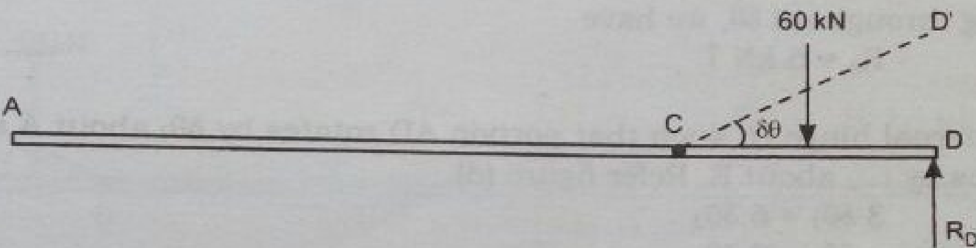
Note that vertical forces being ⊥ to the displacement don't do work.



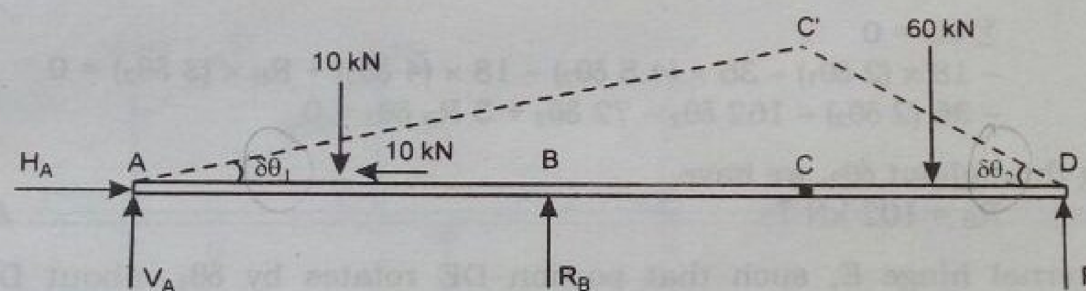
10√2 kN
20 kN/m
A
45°
B
C
D
2.5 m
2.5m
3 m
3 m



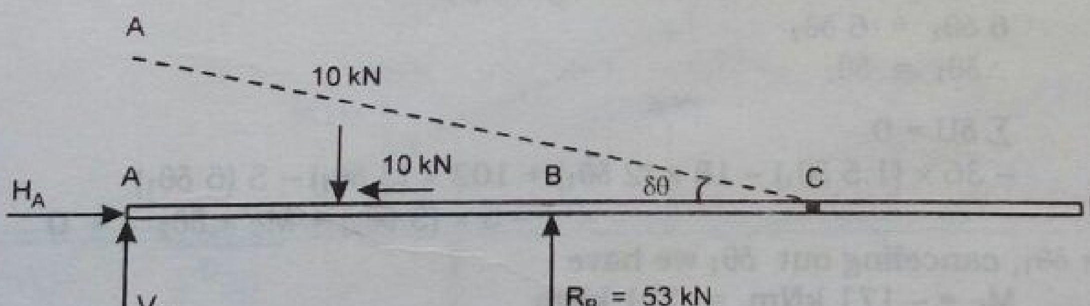
10 kN
60 kN
10 kN
H_A
B
C
D
V_A
R_B
1.5 m
R_D



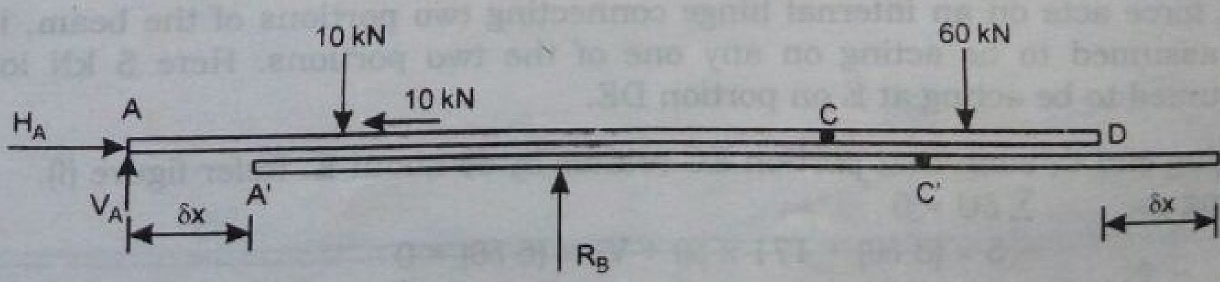
60 kN
D'
A
C
δθ
D
R_D



C'
60 kN
10 kN
10 kN
H_A
A
δθ₁
B
C
δθ₂
D
V_A
R_B
R_D



A
10 kN
10 kN
H_A
A
B
δθ
C
D
V_A
R_B = 53 kN



10 kN
60 kN
10 kN
H_A
A
C
D
D'
V_A
A'
C'
δx
δx
R_B

x. 8.3 Using principle of virtual work, find reactions at A, B and C for the beam shown n figure (a). Note D and E are internal hinges.

olution:

tep 1: There are three beam lengths viz. Portion AD, portion DE and portion EC, internally hinged at D and E. The system has roller support at A and B and fixed support at C. The FBD is shown in figure (b). Since the u.v.l is spread over portion AD and DE it is to be taken as a separate u.v.l. of intensity varying from 0 to 12 kN/m on portion AD and a u.v.l of intensity varying from 12 kN/m to 24 kN/m over DE. This results in a point load of 18 kN on AD and two point loads of 36 kN and 18 kN on DE.

ep 2: Lifting end A, such that portion AD rotates by $\delta\theta$ about D. Refer figure (c).

Using $\Sigma\, \delta U = 0$

$$R_A \times (3\,\delta\theta) - 18 \times (1\,\delta\theta) = 0$$

Canceling throughout $\delta\theta$, we have

$$R_A = 6 \text{ kN} \uparrow$$ .... **Ans.**

ep 3: Lifting internal hinge D, such that portion AD rotates by $\delta\theta_1$ about A and portion DE rotates by $\delta\theta_2$ about E. Refer figure (d).

Here $3\,\delta\theta_1 = 6\,\delta\theta_2$

$\therefore$ $\delta\theta_1 = 2\,\delta\theta_2$

Using $\Sigma\, \delta U = 0$

$$-18 \times (2\,\delta\theta_1) - 36 \times (4.5\,\delta\theta_2) - 18 \times (4\,\delta\theta_2) + R_B \times (3\,\delta\theta_2) = 0$$

$$-36\,(2\,\delta\theta_2) - 162\,\delta\theta_2 - 72\,\delta\theta_2 + 3\,R_B\,\delta\theta_2 = 0$$

Canceling throughout $\delta\theta_2$, we have

$$R_B = 102 \text{ kN} \uparrow$$ .... **Ans.**

ep 4: Lifting internal hinge E, such that portion DE rotates by $\delta\theta_1$, about D an portion EC rotates by $\delta\theta_2$ about C. Refer figure (e).

Here $6\,\delta\theta_1 = 6\,\delta\theta_2$

$\therefore \delta\theta_1 = \delta\theta_2$

Using $\Sigma\, \delta U = 0$

$$-36 \times (1.5\,\delta\theta_1) - 18 \times (2\,\delta\theta_1) + 102 \times (3\,\delta\theta_1) - 5\,(6\,\delta\theta_1) - 5 \times (3\,\delta\theta_2) + M_C \times \delta\theta_2 = 0$$

Since $\delta\theta_2 = \delta\theta_1$, canceling out $\delta\theta_1$ we have

$$M_C = -171 \text{ kNm} = 171 \text{ kNm}$$ .... **Ans.**

e: If a force acts on an internal hinge connecting two portions of the beam, it ca be assumed to be acting on any one of the two portions. Here 5 kN load assumed to be acting at E on portion DE.

5: Lifting end C, such that portion EC rotates by $\delta\theta$ about E. Refer figure (f).

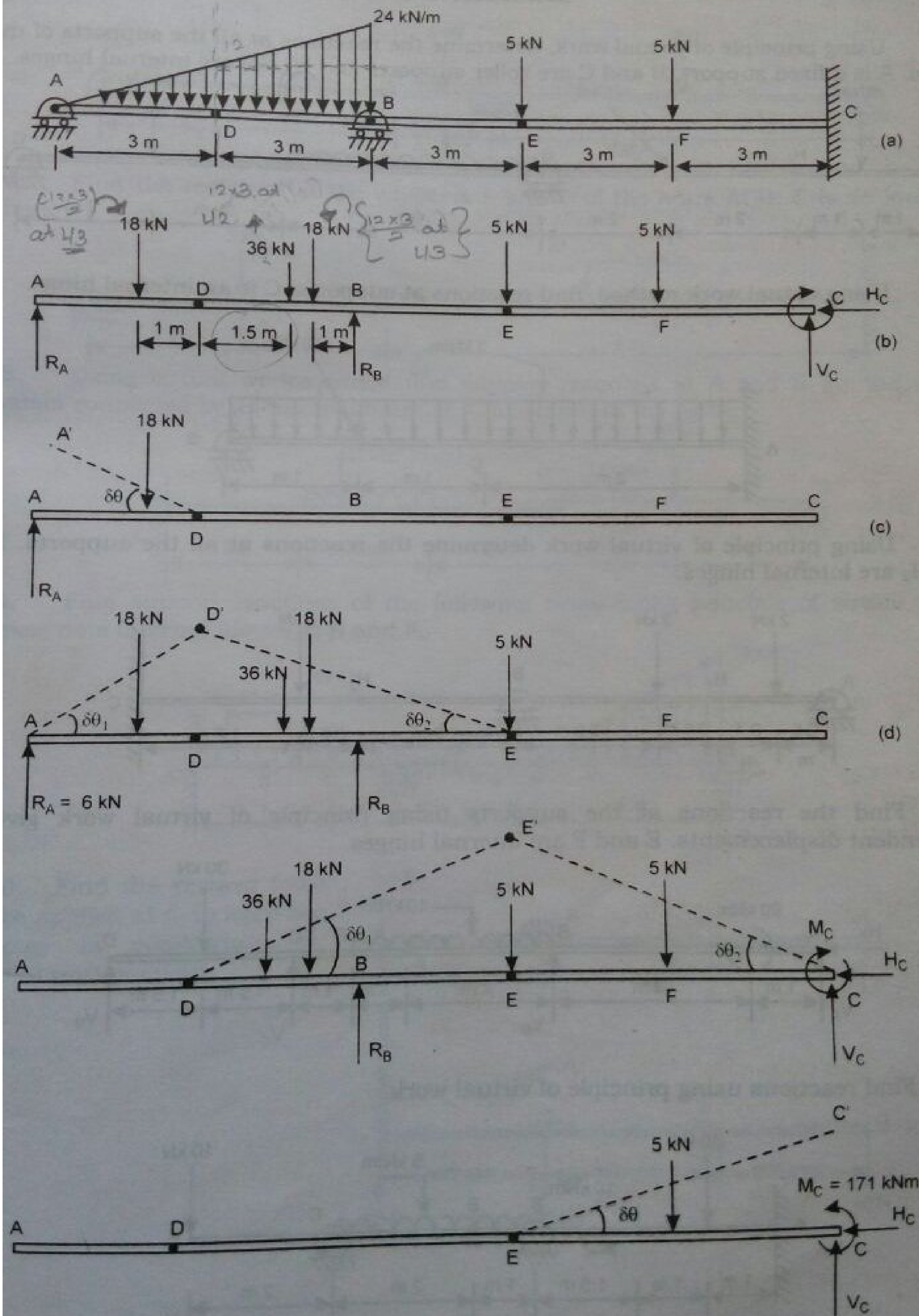
Using $\Sigma\, \delta U = 0$

$$-5 \times (3\,\delta\theta) + 171 \times \delta\theta + V_C \times (6\,\delta\theta) = 0$$

Canceling throughout $\delta\theta$

$\therefore$ $V_C = -26 \text{ kN} = 26 \text{ kN} \downarrow$

.... **Ans**


24 kN/m
5 kN
5 kN
A
B
C
D
E
F
3 m
3 m
3 m
3 m
3 m
(a)
18 kN
36 kN
18 kN
5 kN
5 kN
1 m
1.5 m
1 m
R_A
R_B
H_C
V_C
(b)
18 kN
A'
δθ
(c)
18 kN
D'
18 kN
36 kN
5 kN
$δθ_1$
$δθ_2$
R_A = 6 kN
R_B
(d)
E'
18 kN
36 kN
5 kN
5 kN
$δθ_1$
$δθ_2$
M_C
H_C
R_B
V_C
C'
5 kN
M_C = 171 kNm
δθ
H_C
V_C

Application of Virtual Work to Problems on Mechanism

Mechanism is a system of connected bodies joined by internal hinges having single or multiple degree of freedom (we shall deal with single degree of freedom problems). Single degree of freedom implies that the equilibrium configuration of the mechanism depends on only one variable.

Procedure for analysis

Step 1: Draw the FBD of the mechanism system.

Step 2: Choose a set of co-ordinate axes taking the origin at some fixed point such that the origin remains stationary when the virtual displacement is given to the system.

Step 3: Fix a variable angle 'θ' which describes the equilibrium configuration.

Step 4: Now give a small virtual displacement to the system such that there is an increase in value of θ by a small amount $\delta\theta$.

Step 5: Identify the active forces in the system which do work and then find their co-ordinates in terms of variable θ.

Step 6: Differentiate the co-ordinates w.r.t. θ to get virtual displacement.

Step 7: Write the virtual work equation $\Sigma\ \delta U = 0$ using the sign convention stated below

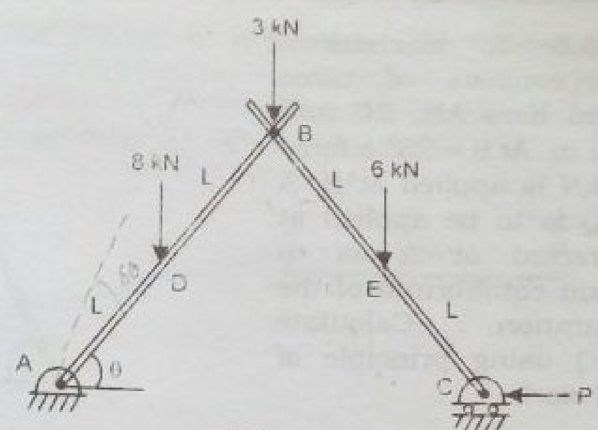
i) If the force acts in the positive direction of the axes it will do positive work, otherwise negative.

ii) If a couple causes an increase in the angle θ it does positive work, otherwise negative.

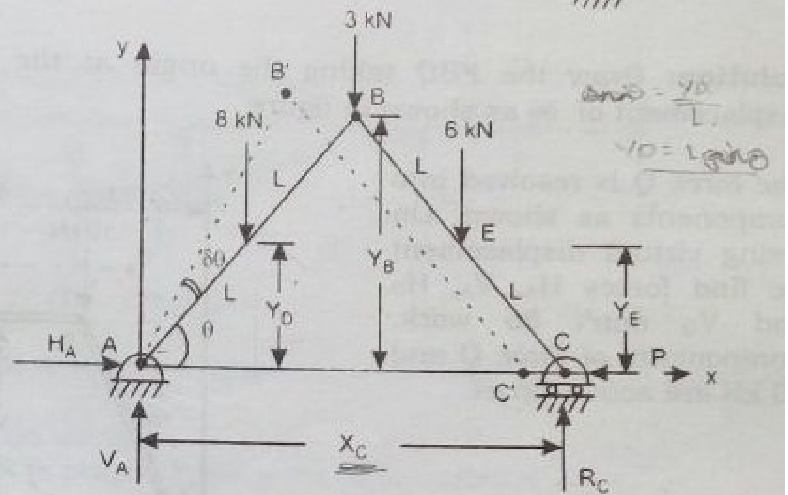
Step 8: Cancel out $\delta\theta$ from the equation. Now substitute the value of θ to get the unknown.

The following examples should help to understand the procedure explained above.

Ex. 8.4 The mechanism shown consists of rods AB and BC each of length 2L hinged at B and externally supported as shown. The system at $\theta = 30°$ is in equilibrium by the application of horizontal force P applied at the roller. Using virtual work method, find the value of P.



Solution: The FBD of the mechanism is shown in figure. Let us take the axes with the origin at hinge A. Giving a virtual displacement such that there is an increase in angle θ by $\delta\theta$ as shown dotted in figure. We find forces H_A, V_A and R_C don't work while forces 8 kN, 3 kN, 6 kN and P are active forces as they do work.



Active Force	Co-ordinate	Virtual Displacement
8 kN	$y_D = L \sin\theta$	$\delta y_D = L \cos\theta \, \delta\theta$
3 kN	$y_B = 2 L \sin\theta$	$\delta y_B = 2 L \cos\theta\delta\theta$
6 kN	$y_E = L \sin\theta$	$\delta y_E = L \cos\theta \, \delta\theta$
P	$x_C = 4 L \cos\theta$	$\delta x_C = -4 L \sin\theta \, \delta\theta$

Using $\Sigma \delta U = 0$

$$-8 \times \delta y_D - 3 \times \delta y_B - 6 \times \delta y_E - P \times \delta x_C = 0$$

$$-8 \times L \cos\theta \, \delta\theta - 3 \times 2 L \cos\theta \, \delta\theta - 6 \times L \cos\theta \, \delta\theta - P \times -4 L \sin\theta \, \delta\theta = 0$$

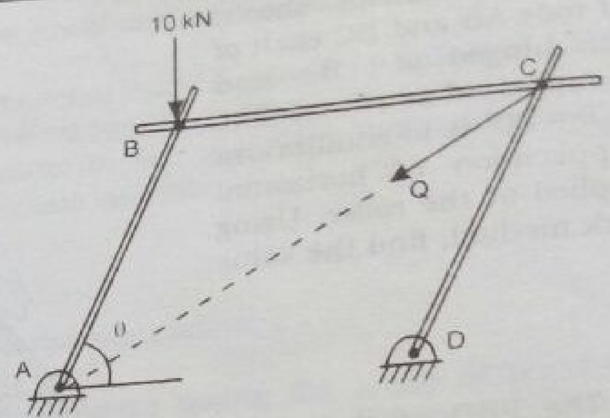
Cancel out L and $\delta\theta$ and substitute $\theta = 30°$

$$-8 \cos 30 - 6 \cos 30 - 6 \cos 30 + 4 P \sin 30 = 0$$

$$\therefore \quad P = 8.66 \text{ kN}$$

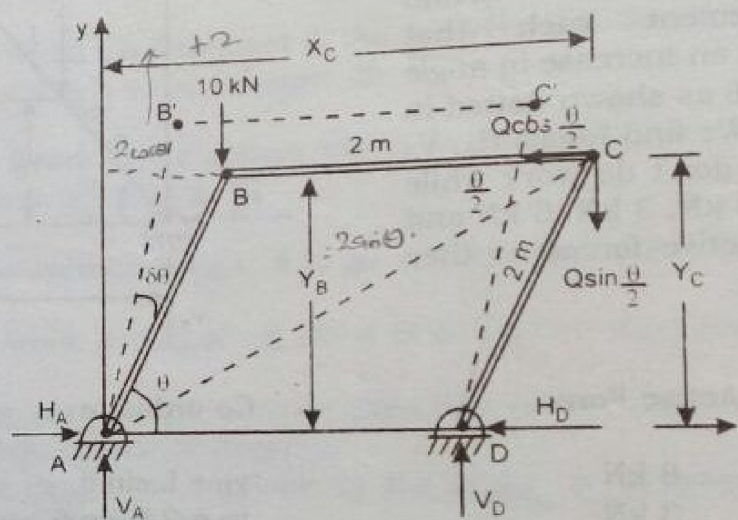
.... **Ans.**

Ex. 8.5 A mechanism shown consists of three identical bars AB, BC and CD = 2 m. At $\theta = 25°$ a force of 10 kN is applied at B. A force Q is to be applied at C, directed at A as to maintain equilibrium of the configuration. Calculate force Q using principle of virtual work.



Solution: Draw the FBD taking the origin at the hinge A. Let us give a vi displacement of $\delta\theta$ as shown in figure

The force Q is resolved into components as shown. On giving virtual displacement we find forces H_A, V_A, H_D and V_D don't do work. Components of force Q and 10 kN are active forces.



Active Force	Co-ordinate	Virtual Displacement
10 kN	$y_B = 2 \sin\theta$	$\delta y_B = 2\cos\theta\,\delta\theta$
$Q\cos(\theta/2)$	$x_C = 2 + 2\cos\theta$	$\delta x_C = -2\sin\theta\,\delta\theta$
$Q\sin(\theta/2)$	$y_C = 2\sin\theta$	$\delta y_C = 2\cos\theta\,\delta\theta$

Using $\Sigma\,\delta U = 0$

$$-10 \times \delta y_B - Q\cos\frac{\theta}{2} \times \delta x_C - Q\sin\frac{\theta}{2} \times \delta y_C = 0$$

$$-10 \times 2\cos\delta\theta - Q\cos\frac{\theta}{2} \times (-2\sin\theta\,\delta\theta) - Q\sin\frac{\theta}{2} \times 2\cos\theta\,\delta\theta = 0$$

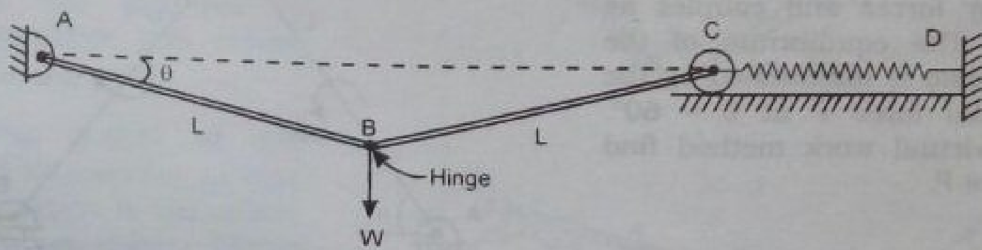
cancel out $\delta\theta$ and substitute $\theta = 25°$

$$-20\cos 25 + Q\cos 12.5 \times 2\sin 25 - Q\sin 12.5 \times 2\cos 25 = 0$$

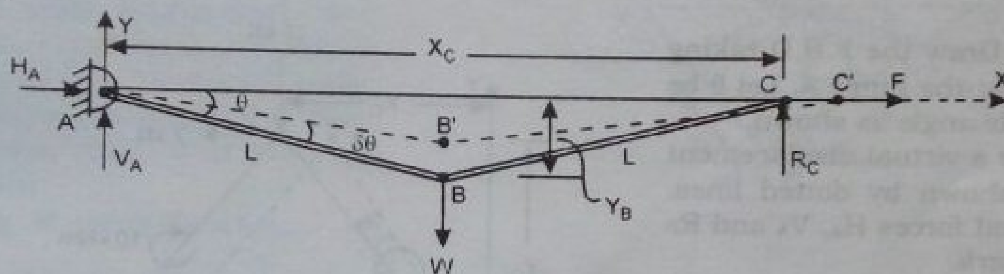
$$\therefore \quad Q = 41.87 \text{ kN}$$

....

Ex. 8.6 A vertical load W is applied to the linkage at B as shown in figure (a). The constant of spring is k and the spring is unstretched when AB and BC are horizontal. Neglecting the weight of the linkage, find the equation in θ, W and L and k which must be satisfied when the linkage is in equilibrium. Use virtual work method.



Solution:



Taking the origin at A. Let us give a virtual displacement of δθ as shown in figure by dotted lines.

We find forces H_A, V_A and R_C don't do work.

We also know that force in spring F is proportional to its deformation x

i.e. $F = k \times x$

since the initial length $AC = 2L$ becomes $AC' = 2L\cos\theta$

∴ the extension of spring $x = 2L - 2L\cos\theta$

∴ $F = k\,(2L - 2L\cos\theta)$

$F = 2\,k\,L\,(1 - \cos\theta)$

Active force	Co-ordinate	Virtual Displacement
W	$y_B = -L\sin\theta$	$\delta y_B = -L\cos\theta\,\delta\theta$
F	$x_C = 2L\cos\theta$	$\delta x_C = -2L\sin\theta\,\delta\theta$

Using $\Sigma\,\delta U = 0$

$- W\,\delta y_B + F \times \delta x_C = 0$

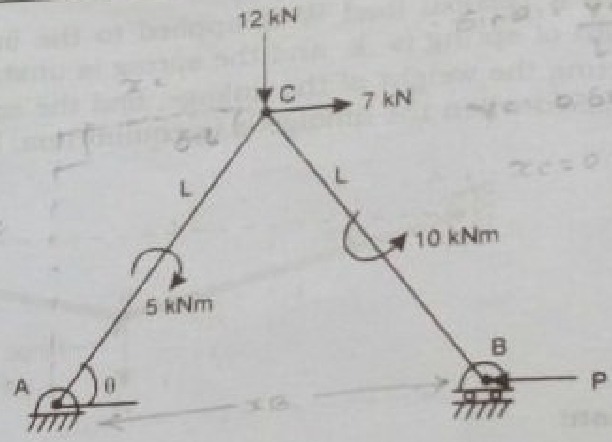
$- W \times (-L\cos\theta\,\delta\theta) + 2\,k\,L\,(1 - \cos\theta) \times (-2L\sin\theta\,\delta\theta) = 0$

cancel out δθ and L

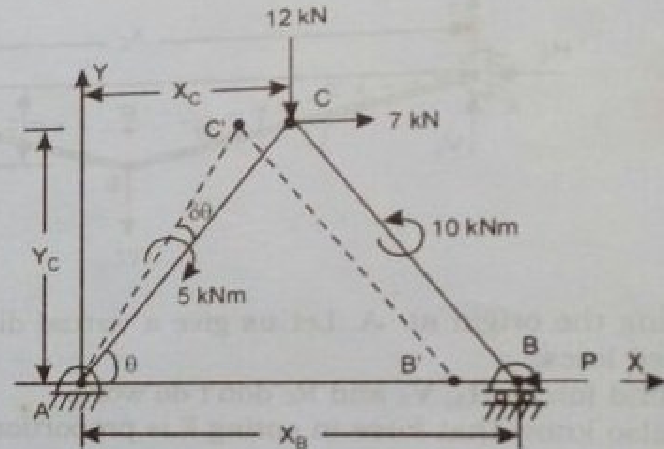
$W\cos\theta - 4\,k\,L\sin\theta\,(1 - \cos\theta) = 0$

or $W = 4\,k\,L\tan\theta\,(1 - \cos\theta)$ ....**Ans.**

Ex. 8.7 A two body mechanism formed by members AC and BC both length L = 0.6 m are acted upon by forces and couples as shown. The equilibrium of the mechanism is maintained by the horizontal force P at $\theta = 60°$. Using virtual work method find the force P.



Solution: Draw the F.B.D taking the origin at the hinge A. Let θ be the variable angle as shown.
Let us give a virtual displacement of δθ as shown by dotted lines.
We also find forces H_A, V_A and R_B don't do work.
We also find that forces 12 kN, 7 kN and P are active forces. Also couples of 5 kNm and 10 kNm are active since they do work.



Active Force	Co-ordinate	Virtual Displacement
12 kN	$y_C = 0.6 \sin\theta$	$\delta y_C = 0.6 \cos\theta\, \delta\theta$
7 kN	$x_C = 0.6 \cos\theta$	$\delta y_B = -0.6 \sin\theta\, \delta\theta$
P	$x_B = 1.2 \cos\theta$	$\delta y_E = -1.2 \sin\theta\, \delta\theta$

Also the couples of 5 kNm and 10 kNm do negative work, since they tend to cause a decrease of angle θ.

Using $\Sigma\, \delta U = 0$

$$-5\,\delta\theta - 10\,\delta\theta - 12\,\delta y_C + 7\,\delta x_C - P\,\delta x_B = 0$$

$$-5\,\delta\theta - 10\,\delta\theta - 12\,[\,0.6\cos\theta\,\delta\theta\,] + 7\,[\,-0.6\sin\theta\,\delta\theta\,] - P\,[\,-1.2\sin\theta\,\delta\theta\,] = 0$$

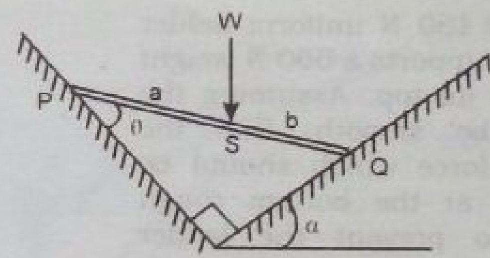
cancel out δθ and substitute $\theta = 60°$

$$-5 - 10 - 7.2\cos 60 - 4.2\sin 60 + 1.2\,P\sin 60 = 0$$

$$\therefore \quad P = 21.4 \text{ kN}$$

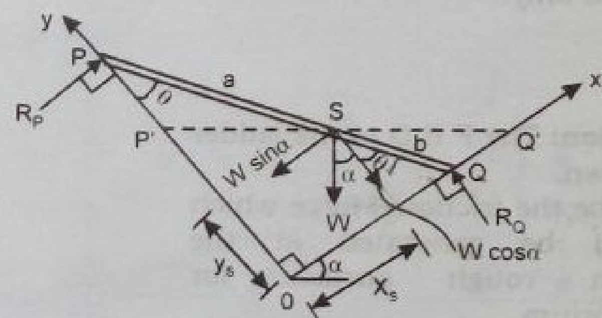
.... **Ans.**

Ex. 8.8 A weightless bar PQ is held in equilibrium between two smooth surfaces as shown. It is acted upon by a load W. Find the angle θ the bar acquires in equilibrium position. Use virtual work method.



Solution: The F.B.D of the mechanism is shown. Let us take the axes with origin at the corner of the surfaces as shown. Giving a virtual displacement such that there is an increase in the angle θ by δθ as shown by dotted lines. We find forces R_P and R_Q are non-active forces while W is the only active force .

Resolving W along the axes.



Active Force	Co-ordinate	Virtual Displacement
$W\cos\alpha$	$y_S = b\cos\theta$	$\delta y_S = -\,b\sin\theta\,\delta\theta$
$W\sin\alpha$	$x_S = a\sin\theta$	$\delta x_S = a\cos\theta\,\delta\theta$

Using $\sum \delta U = 0$

$$-W\cos\alpha \times \delta y_S - W\sin\alpha \times \delta x_S = 0$$

$$-W\cos\alpha \times (-\,b\sin\theta\,\delta\theta) - W\sin\alpha \times (a\cos\theta\,\delta\theta) = 0$$

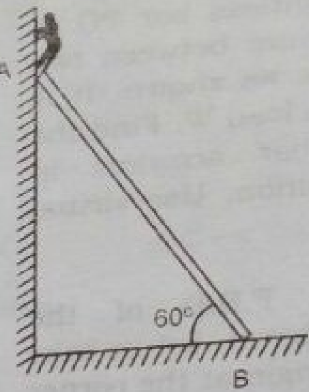
cancel out δθ and W, we get

$$b\cos\alpha\sin\theta - a\sin\alpha\cos\theta = 0$$

$$\tan\theta = \frac{a}{b}\tan\alpha$$

$$\therefore \quad \theta = \tan^{-1}\left(\frac{a}{b}\tan\alpha\right) \quad \text{............ Ar}$$

Ex. 8.9 A 150 N uniform ladder 4 m long supports a 500 N weight person at its top. Assuming the wall to be smooth, find the frictional force which should be generated at the bottom rough surface to prevent the ladder from slipping. Use virtual work method only.



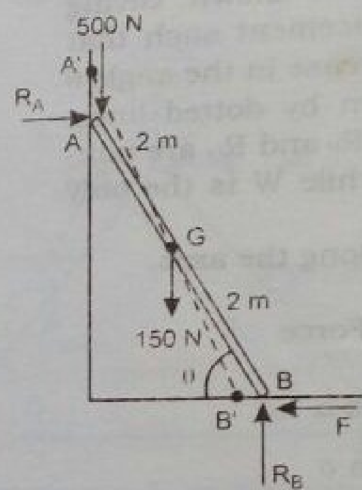
Solution: The F.B.D of the ladder is shown.

Let F be the frictional force which should be generated at the bottom rough surface for equilibrium.

Taking the co-ordinate axes, origin and variable angle θ as shown in figure.

Let us give a virtual displacement as shown by dotted lines.

We find forces R_A, R_B are non-active forces while 150 N, 500 N and F are active forces.



Active Force	Co-ordinate	Virtual Displacement
150 kN	$y_G = 2 \sin \theta$	$\delta y_G = 2 \cos \theta \, \delta\theta$
500 kN	$y_A = 4 \sin \theta$	$\delta y_A = 4 \cos \theta \, \delta\theta$
F	$x_B = 4 \cos \theta$	$\delta x_B = -4 \sin \theta \, \delta\theta$

Using $\sum \delta U = 0$

$$-150\, \delta y_G - 500\, \delta y_A - F\, \delta x_B = 0$$

$$-150\,(2 \cos \theta\, \delta\theta) - 500\,(4 \cos \theta\, \delta\theta) - F\,(-4 \sin \theta\, \delta\theta) = 0$$

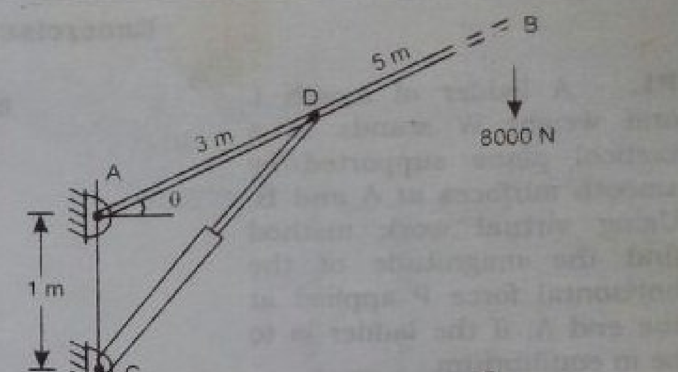
cancel out δθ and substitute θ = 60°

$$-300 \cos 60 - 2000 \cos 60 + 4\,F \sin 60 = 0$$

$$\therefore \quad F = 331.97 \text{ N}$$

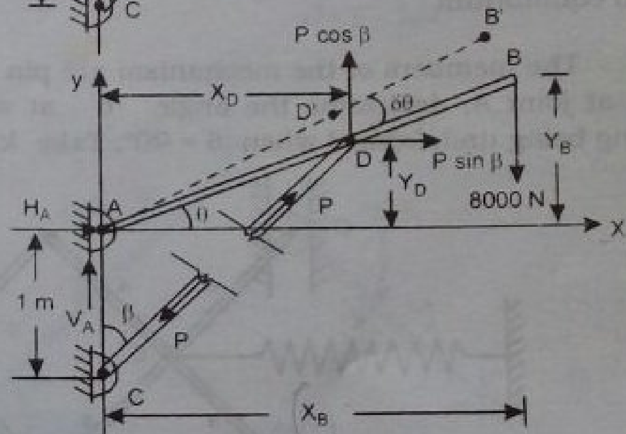
.... **An**

Ex. 8.10 The boom AB of a hoisting crane is hinged at A and is supported and adjusted by a hydraulic cylinder CD pinned at D. At a position $\theta = 30°$, determine the force developed in the hydraulic cylinder while hoisting a load of 8000 N.



Solution: Cutting member CD.

Let P be the magnitude of the force in the hydraulic cylinder CD. Let the nature of the force be compressive.



From geometry of Δ ACD

Using cosine rule L (CD)

$$= \sqrt{3^2 + 1^2 - 2 \times 3 \times 1 \cos 120}$$

$$= 3.605 \text{ m}$$

Using sine rule $\dfrac{3}{\sin\beta} = \dfrac{3.605}{\sin 120}$

$\therefore \quad \beta = 46.1°$

Taking the coordinate axes as shown with the origin at hinge A.

Let θ be the variable angle as shown in figure. Let us give a virtual displacement as shown by dotted lines.

Force H_A and V_A are non active forces, while forces P and 8000 N are active forces. Resolving P into components P cos β and P sin β.

Active Force	Co-ordinate	Virtual Displacement
8000 N	$y_B = 8 \sin\theta$	$\delta y_B = 8 \cos\theta\, \delta\theta$
P cos β	$y_D = 3 \sin\theta$	$\delta y_D = 3 \cos\theta\, \delta\theta$
P sin β.	$x_D = 3 \cos\theta$	$\delta x_D = -3 \sin\theta\, \delta\theta$

Using $\Sigma\, \delta U = 0$

$$-8000.\delta y_B + P\cos\beta.\delta y_D + P\sin\beta.\delta x_D = 0$$

$$-\ 8000.(8\cos\theta\,\delta\theta) + P\cos\beta.(3\cos\theta\,\delta\theta) + P\sin\beta\,(-3\sin\theta\,\delta\theta) = 0$$

Canceling out $\delta\theta$ and substituting $\theta = 30°$, $\beta = 46.1°$, we get

P = 76907 N (compressive) .... **Ans.**